Coherent Sheaves on Singular Projective Curves with Nodal Singularities

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Abstract

We give the full answer to the question: on which curves the category of coherent sheaves Coh_X is tame. The answer is: these are just the curves from the list of Drozd-Greuel (see [6]). Moreover, in this case the derived category $D^-(\mathrm{Coh}_X)$ is also tame. We give an explicit description of the objects of this category as well as of the categories $D^b(\mathrm{Coh}_X)$, Coh_X. Among the coherent sheaves we describe the vector bundles, torsion-free sheaves, mixed sheaves and skyscraper sheaves.

1   Introduction

Let X be a projective curve over $k = \bar{k}$. For any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$ we have
$$\dim_k(\mathrm{Hom}(\mathcal{F}, \mathcal{G})) < \infty.$$

This implies that in the category of coherent sheaves the generalized Krull-Schmidt theorem holds:
$$\mathcal{F} \cong \bigoplus_{i=1}^{s} \mathcal{F}_i^{m_i},$$

where $\mathcal{F}_i$ are indecomposable and $m_i, \mathcal{F}_i$ are uniquely defined.
<u>Our aim</u>: to describe all indecomposable coherent sheaves on X. Consider first a smooth case. For any coherent sheaf $\mathcal{F}$ we have

$$0 \longrightarrow T(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/T(\mathcal{F}) \longrightarrow 0,$$

where $T(\mathcal{F})$ is the torsion part of $\mathcal{F}$ (skyscraper sheaf) and $\mathcal{F}/T(\mathcal{F})$ the torsion-free quotient of $\mathcal{F}$. But $\mathcal{F}/T(\mathcal{F})$ is even locally free, since our curve is smooth. The local-global spectral sequence $H^p(\mathcal{E}xt^q(\ ,\ )) \Longrightarrow \mathrm{Ext}^{p+q}(\ ,\ )$ implies

$$0 \longrightarrow H^1(\mathcal{H}om(TF(\mathcal{F}/T(\mathcal{F}), T(\mathcal{F})) \longrightarrow \mathrm{Ext}^1(\mathcal{F}/T(\mathcal{F}), T(\mathcal{F})) \longrightarrow H^0(\mathcal{E}xt^1(\mathcal{F}/T(\mathcal{F}), T(\mathcal{F})) \longrightarrow 0.$$

So, $\mathrm{Ext}^1(\mathcal{F}/T(\mathcal{F}), T(\mathcal{F})) = 0$ and $\mathcal{F} \cong T(\mathcal{F})) \oplus \mathcal{F}/T(\mathcal{F})$. Also, in a smooth case, indecomposable objects of Coh_X are

- scyscraper sheaves $\mathcal{O}_x/m_x^n$

- indecomposable vector bundles.

What is known about the classification of indecomposable vector bundles on smooth projective curves?

1. Let $X = \mathbf{P}^1_k$. Then indecomposable vector bundles are just the line bundles $\mathcal{O}_{\mathbf{P}^1}(n), n \in \mathbb{Z}$ (see [10]).

2. Let X be an elliptic curve. The indecomposable vector bundles are described by two discrete parameters r, d: rank and degree and one continuous (point of the curve X) [1].

3. It is well-known that with the growth of the genus of the curve $g(X)$ the moduli spaces of vector bundles become bigger and bigger. For the smooth curves of genus $g \geq 2$ it was shown (Drozd/Greuel [6], Scharlau(1992)) that the problem of classification of vector bundles is wild. "Wild" means

   (a) "geometrically": we have n-parameter families of indecomposable vector bundles for arbitrary large n;

   (b) "algebraically": for every finite-dimensional k-algebra Λ there is an exact functor $(\Lambda - \mathrm{mod}) \longrightarrow VB_X$ mapping non-isomorphic objects to non-isomorphic ones and indecomposable into indecomposable.

Moreover, Drozd and Greuel have proved the following trichotomy (see [6]):

1. VB_X is finite (indecomposable objects are described by discrete parameters) if X is a configuration of projective lines of the type A_n.
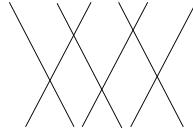

2. VB_X is tame (intuitively this means that indecomposable objects are parametrized by 1 continuous parameter and several discrete parameters. For the rigorous definition see also [6]) if

   (a) X is an elliptic curve
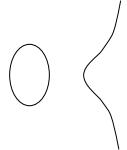
(b) X is a rational curve with one simple node
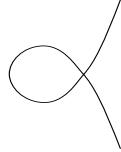
(c) X is a configuration of projective lines of type $\tilde{A}_n$
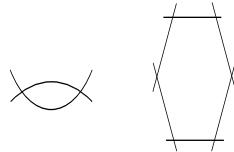
3. Wild, otherwise.

Drozd and Greuel also gave an explicit description of vector bundles and torsion-free sheaves. We want to describe the coherent sheaves in the cases $1 - 2$. The problem is that the sequence

$$0 \longrightarrow T(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/T(\mathcal{F}) \longrightarrow 0$$

does not necessarily split. ($\mathcal{F}/T(\mathcal{F})$ could be torsion-free but not locally free.) We can descibe all skyscraper sheaves (see [12], [5], [9]) and torsion-free sheaves. But there is no common technology of reconstructing the objects possibly being at the middle. We need a new idea and this idea is to apply the technique of the derived categories. There is a full and faithful functor $\mathrm{Coh}_X \hookrightarrow D^-(\mathrm{Coh}_X)$ sending a coherent sheaf $\mathcal{F}$ into its locally free resolution $\mathcal{F}_{\cdot}$. We shall describe indecomposable objects of the bigger category $D^-(\mathrm{Coh}_X)$ and among them the complexes with zero higher homologies (which correspond to coherent sheaves).

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2   Main construction

Let X be a projective curve of the type as in the list of Drozd-Greuel (singular points are simple double points or transversal intersections), $\tilde{X} \xrightarrow{\pi} X$ its normalization, $\tilde{\mathcal{O}} = \pi_*(\mathcal{O}_{\tilde{X}})$ (so $\tilde{\mathcal{O}}_x = \bar{\mathcal{O}}_{X,x}$ — the integral closure of $\mathcal{O}_{X,x}$), $\mathcal{J} = Ann_{\mathcal{O}}(\tilde{\mathcal{O}}/\mathcal{O})$ the conductor (so the support of $\mathcal{O}/\mathcal{J}$ is precisely the set of singular points $Sing(X)$). Since the morphism π is affine, we can identify $\mathcal{O}_{\tilde{X}}$-modules and $\tilde{\mathcal{O}}$-modules.

Definition 2.1 *Consider the following category of triples of complexes* TC_X *(for convenience, all objects of the derived categories below are supposed to be the complexes of locally-free modules)*

1. *Objects are the triples* $(\tilde{\mathcal{F}}_\cdot, \mathcal{M}_\cdot, i)$, *where*

   $\tilde{\mathcal{F}}_\cdot \in D^-(\mathrm{Coh}_{\tilde{\mathcal{O}}})$,

   $\mathcal{M}_\cdot \in D^-(\mathrm{Coh}_{\mathcal{O}/\mathcal{J}})$,

   $i : \mathcal{M}_\cdot \longrightarrow \tilde{\mathcal{F}}_\cdot \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{O}}/\mathcal{J}$ *(* $\otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{O}}/\mathcal{J}$ *denotes the derived functor of the tensor product) a morphism in* $D^-(\mathrm{Coh}_{\mathcal{O}/\mathcal{J}})$, *such that*

   $i \otimes id : \mathcal{M}_\cdot \otimes_{\mathcal{O}} \tilde{\mathcal{O}}/\mathcal{J} \longrightarrow \tilde{\mathcal{F}}_\cdot \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{O}}/\mathcal{J}$ *is an isomorphism in* $D^-(\mathrm{Coh}_{\tilde{\mathcal{O}}/\mathcal{J}})$ *(we implicitly use here that* $gl.dim(\mathrm{Coh}_{\mathcal{O}/\mathcal{J}}) = 0$. *The last means that the morphism* $i \otimes id$ *is correctly defined). We want to stress (it is the only exception to the agreement above) that* $i \otimes id$ *is not the derived functor of the tensor product but just the tensor product.*

2. *Morphisms* $(\tilde{\mathcal{F}}_{1\cdot}, \mathcal{M}_{1\cdot}, i_1) \longrightarrow (\tilde{\mathcal{F}}_{2\cdot}, \mathcal{M}_{1\cdot}, i_2)$ *are pairs* (Φ, φ), $\tilde{\mathcal{F}}_{1\cdot} \xrightarrow{\Phi} \tilde{\mathcal{F}}_{2\cdot}, \mathcal{M}_{1\cdot} \xrightarrow{\phi} \mathcal{M}_{2\cdot}$, *such that*

$$
\begin{array}{ccccc}
\tilde{\mathcal{F}}_{1\cdot} & \longrightarrow & \tilde{\mathcal{F}}_{1\cdot} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{O}}/\mathcal{J} & \xleftarrow{i_1} & \mathcal{M}_{1\cdot} \\
\downarrow{\Phi} & & \downarrow{\Phi \otimes id} & & \downarrow{\phi} \\
\tilde{\mathcal{F}}_{2\cdot} & \longrightarrow & \tilde{\mathcal{F}}_{2\cdot} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{O}}/\mathcal{J} & \xleftarrow{i_2} & \mathcal{M}_{2\cdot}
\end{array}
$$

   is commutative (more precisely, we want the right square to be commutative in $D^-(\mathrm{Coh}_{\mathcal{O}/\mathcal{J}})$).

Theorem 2.2 *The functor*

$$
D^-(\mathrm{Coh}_X) \xrightarrow{\mathbf{F}} \mathrm{TC}_X
$$

$\mathcal{F}^\cdot \longrightarrow (\mathcal{F}_\cdot \otimes_{\mathcal{O}} \tilde{\mathcal{O}}, \mathcal{F}_\cdot \otimes_{\mathcal{O}} \mathcal{O}/\mathcal{J}, i : \mathcal{F}_\cdot \otimes_{\mathcal{O}} \mathcal{O}/\mathcal{J} \longrightarrow \mathcal{F}_\cdot \otimes_{\mathcal{O}} \tilde{\mathcal{O}}/\mathcal{J})$ *fulfils the following properties:*

1. $\mathbf{F}$ *is dense (i.e., every triple* $(\tilde{\mathcal{F}}_\cdot, \mathcal{M}_\cdot, i)$ *is isomorphic to some* $\mathbf{F}(\mathcal{F}_\cdot)$).

2. $\mathbf{F}(\mathcal{F}_\cdot) \cong \mathbf{F}(\mathcal{G}_\cdot) \Longleftrightarrow \mathcal{F}_\cdot \cong \mathcal{G}_{\cdot\cdot}$.

3. $\mathbf{F}$ *is full.*

Remark 2.3 $\mathbf{F}$ *is not faithful. So it is not an equivalence.*

Proof. The main ingredient of the proof is: having a triple $(\tilde{\mathcal{F}}_\cdot, \mathcal{M}_\cdot, i)$ how can we reconstruct $\mathcal{F}_\cdot$?

The exact sequence

$$0 \longrightarrow \mathcal{J}\tilde{\mathcal{F}}_{\cdot} \longrightarrow \tilde{\mathcal{F}}_{\cdot} \longrightarrow \tilde{\mathcal{F}}_{\cdot} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{O}}/\mathcal{J} \longrightarrow 0$$

in Coh_X gives a distinguished triangle

$$\mathcal{J}\tilde{\mathcal{F}}_{\cdot} \longrightarrow \tilde{\mathcal{F}}_{\cdot} \longrightarrow \tilde{\mathcal{F}}_{\cdot} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{O}}/\mathcal{J} \longrightarrow \mathcal{J}\tilde{\mathcal{F}}_{\cdot}[-1]$$

in $D^-(\mathrm{Coh}_X)$. The properties of the triangulated categories imply the morphism of triangles

$$
\begin{array}{ccccccc}
\mathcal{J}\tilde{\mathcal{F}}_{\cdot} & \longrightarrow & \tilde{\mathcal{F}}_{\cdot} & \longrightarrow & \tilde{\mathcal{F}}_{\cdot} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{O}}/\mathcal{J} & \longrightarrow & \mathcal{J}\tilde{\mathcal{F}}_{\cdot}[-1] \\
\uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \Phi} & & \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle id} \\
\mathcal{J}\tilde{\mathcal{F}}_{\cdot} & \longrightarrow & \mathcal{F}_{\cdot} & \longrightarrow & \mathcal{M}_{\cdot} & \longrightarrow & \mathcal{J}\tilde{\mathcal{F}}_{\cdot}[-1].
\end{array}
$$

In other words $\mathcal{F}_{\cdot} = cone(\mathcal{M}_{\cdot} \longrightarrow \mathcal{J}\mathcal{F}_{\cdot}[-1])[1]$. Taking a cone is not a functorial operation. It gives an intuitive explanation, why functor $\mathbf{F}$ is not an equivalence.

The properties of the triangulated categories imply immediately that the constructed map

$$\mathbf{G} : Ob(\mathrm{TC}_X) \longrightarrow Ob(D^-(\mathrm{Coh}_X))$$

sends isomorphic objects into isomorphic ones and satisfies $\mathbf{GF}(\mathcal{F}_{\cdot}) \cong \mathcal{F}_{\cdot}$. Now we have to show, that $\mathbf{FG}(\tilde{\mathcal{F}}_{\cdot}, \mathcal{M}_{\cdot}, i) \cong (\tilde{\mathcal{F}}_{\cdot}, \mathcal{M}_{\cdot}, i)$. Suppose we have found a representative of $\mathcal{M}_{\cdot}$ (in the corresponding class of the derived category) such that $i \otimes id$ is an isomorphism *of complexes*. Consider the pull-back diagram in the abelian category $Com(\mathrm{Coh}_X)$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{J}\tilde{\mathcal{F}}_{\cdot} & \longrightarrow & \mathcal{F}_{\cdot} & \overset{\Psi}{\longrightarrow} & \mathcal{M}_{\cdot} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & \mathcal{J}\tilde{\mathcal{F}}_{\cdot} & \longrightarrow & \tilde{\mathcal{F}}_{\cdot} & \overset{\pi}{\longrightarrow} & \tilde{\mathcal{F}}_{\cdot}/\mathcal{J}\tilde{\mathcal{F}}_{\cdot} & \longrightarrow & 0.
\end{array}
$$

We are going to show that

1. $\mathcal{F}_{\cdot}$ is a complex of locally-free $\mathcal{O}$-modules;

2. $(\Phi \otimes id, \Psi \otimes id) : (\mathcal{F}_{\cdot} \otimes_{\mathcal{O}} \tilde{\mathcal{O}}, \mathcal{F}_{\cdot} \otimes_{\mathcal{O}} \mathcal{O}/\mathcal{J}, \mathcal{F}_{\cdot} \otimes_{\mathcal{O}} \mathcal{O}/\mathcal{J} \longrightarrow \mathcal{F}_{\cdot} \otimes_{\mathcal{O}} \tilde{\mathcal{O}}/\mathcal{J}) \longrightarrow (\tilde{\mathcal{F}}_{\cdot}, \mathcal{M}_{\cdot}, i)$ is an isomorphism in the category of triples.

The first condition should be checked in each stalk of every component of the complex $\mathcal{F}_{\cdot}$. We have to show that $\forall x \in X, n \geq 0 \; (\mathcal{F}_n)_x$ is a locally-free $\mathcal{O}_{X,x}$-module. In a regular point it is obvious. Let x be a singular point of a curve. Denote $A = \mathcal{O}_{X,x}, \tilde{A} = \tilde{\mathcal{O}}_{X,x}, J = \mathcal{J}_x, F = (\mathcal{F}_n)_x, \tilde{F} = (\tilde{\mathcal{F}}_n)_x, M = (\mathcal{M}_n)_x$. From the projectivity of A follows a commutative diagram:

$$
\begin{array}{c}
A^r \\
\end{array}
$$

$$
\xymatrix{
& & & A^r \ar[d] \ar[dll]_{\psi} & \\
& & & (A/J)^r \ar[d]^{iso} & \\
0 \ar[r] & J\tilde{F} \ar[r] \ar[d]^{id} & F \ar[r]^{\Psi} \ar[d]^{\Phi} & M \ar[r] \ar[d]^{i} & 0 \\
0 \ar[r] & J\tilde{F} \ar[r] & \tilde{F} \ar[r] & \tilde{F}/J\tilde{F} \ar[r] & 0.
}
$$

Since Φ is injective, F is a torsion-free module. Tensor the first row of the exact sequence with $\otimes_A \tilde{A}/J$ and the second with $\otimes_{\tilde{A}} \tilde{A}/J$. Since $i \otimes id$ is an isomorphism, we obtain that $\Phi \otimes id$ is an isomorphism modulo the radical. By Nakayama's lemma, $\Phi \otimes id$ is an epimorphism. By the same reason ψ is also an epimorphism. But $rank(F) = rank(\tilde{F}) = r$. ψ is an epimorphism of torsion-free modules of the same rank. So, ψ is an isomorphism. The same holds for $\Phi \otimes id$. We have shown, moreover, that $\Phi \otimes id$ is an isomorphism of complexes. The same we can say about $\Psi \otimes id$. Now let us prove the statement we have used.

Statement 2.4 *Let $\tilde{\mathcal{F}}_.$ be a complex of locally-free $\tilde{\mathcal{O}}$-modules, $i : \mathcal{M}_. \longrightarrow \tilde{\mathcal{F}}_./\mathcal{J}$ a morphism in $D^-(\mathrm{Coh}_{\mathcal{O}/\mathcal{J}})$ such that $i \otimes id$ is an isomorphism in $D^-(\mathrm{Coh}_{\tilde{\mathcal{O}}/\mathcal{J}})$. Then there is a complex $\mathcal{M}'$ and a homotopy isomorphism $f : \mathcal{M}'_. \longrightarrow \mathcal{M}_.$ such that for $i' = if$ the map $i' \otimes id : M' \otimes_{\mathcal{O}/\mathcal{J}} \tilde{\mathcal{O}}/\mathcal{J} \longrightarrow \tilde{\mathcal{F}}_./\mathcal{J}$ is an isomorphism of complexes.*

Proof. Let us mention the following easy

Lemma 2.5 *Let $\mathcal{A}$ be an abelian category of a homological dimension 0, $X_.$ and $Y_.$ two complexes, $f_. : X_. \longrightarrow Y_.$ a morphism of complexes. Then the following holds:*

1. *Let $B_n(X) = Im(d_{n+1}(X_.))$. Then $X_n = B_n(X) \oplus H_n(X) \oplus B_{n-1}(X)$.*

2. *Let $d_n : B_n(X) \oplus H_n(X) \oplus B_{n-1}(X) \longrightarrow B_{n-1}(X) \oplus H_{n-1}(X) \oplus B_{n-2}(X)$ be given by $d_n(b_n, h_n, b_{n-1}) = (b_{n-1}, 0, 0)$. It defines a complex, quasi-isomorphic to $X_.$.*

3. *In such a notation the map $f_.$ looks as follows:*
$$
f_n = \begin{pmatrix} f_n|_{B_n(X)} & 0 & 0 \\ 0 & H_n(f) & 0 \\ 0 & 0 & f_{n-1}|_{B_{n-1}(X)} \end{pmatrix}.
$$

4. *Moreover, the morphism of complexes*
$$
\tilde{f}_n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & H_n(f) & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

6

is homotopic to $f_.$. We'll call it also a canonical form of a morphism f.

Suppose X is a curve with simple nodes. Consider complex $\tilde{\mathcal{F}}_./\mathcal{J}$ from $D^-(\mathrm{Coh}_{\tilde{\mathcal{O}}/\mathcal{J}})$. We see that all $(\tilde{\mathcal{F}}_n/\mathcal{J})_x$ (x is a singular point of X) considered as k-vector spaces are even-dimensional. Moreover, all homologies $H_n((\tilde{\mathcal{F}}_./\mathcal{J})_x)$ are even-dimensional, too. So, all the boundaries $B_n((\tilde{\mathcal{F}}_./\mathcal{J})_x)$ are also even-dimensional. The proof is an easy linear algebra now, so we shall leave it. We have shown that our functor $\mathbf{F}$ is dense and is a bijection on the iso-classes of indecomposable objects. Let us show that $\mathbf{F}$ is full.

Let $(\Phi,\phi) : (\tilde{\mathcal{F}}_{1.},\mathcal{M}_{1.},i_1) \longrightarrow (\tilde{\mathcal{F}}_{2.},\mathcal{M}_{2.},i_2)$ be a morphism in TC_X, where i_1,i_2 induce an isomorphism of complexes after tensoring, Φ *is a morphism of complexes*. If both ϕ and $\bar{\Phi}$ are in the canonical form, then the following diagram

$$
\begin{array}{ccc}
\mathcal{M}_{1.} & \xrightarrow{\ \phi\ } & \mathcal{M}_{2.} \\
\downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} \\
\tilde{\mathcal{F}}_{1.}/\mathcal{J}\tilde{\mathcal{F}}_{1.} & \xrightarrow{\ \bar{\Phi}\ } & \tilde{\mathcal{F}}_{2.}/\mathcal{J}\tilde{\mathcal{F}}_{2.}
\end{array}
$$

is commutative *in the category of complexes*. The properties of pull-back imply the existence of a morphism of complexes $\mathcal{F}_{1.} \longrightarrow \mathcal{F}_{2.}$ such that

$$
\begin{array}{c}
\mathcal{F}_{2.} \longrightarrow \mathcal{M}_{2.} \\
\mathcal{F}_{1.} \longrightarrow \mathcal{M}_{1.} \\
\tilde{\mathcal{F}}_{2.} \longrightarrow \tilde{\mathcal{F}}_{2.}/\mathcal{J}\tilde{\mathcal{F}}_{2.} \\
\tilde{\mathcal{F}}_{1.} \longrightarrow \tilde{\mathcal{F}}_{1.}/\mathcal{J}\tilde{\mathcal{F}}_{1.}
\end{array}
$$

is commutative. If $\Phi : \tilde{\mathcal{F}}_{1.} \longrightarrow \tilde{\mathcal{F}}_{1.}$ is a quasi-isomorphism, then $\bar{\Phi} : \tilde{\mathcal{F}}_{1.}/\mathcal{J}\tilde{\mathcal{F}}_{1.} \longrightarrow \tilde{\mathcal{F}}_{2.}/\mathcal{J}\tilde{\mathcal{F}}_{2.}$ is a quasi-isomorphism, too (the tensor product is a functor). The axioms of triangulated categories imply that $\mathcal{J}\tilde{\mathcal{F}}_. \longrightarrow \mathcal{J}\tilde{\mathcal{G}}_.$ is also a quasi-isomorphism. Hence, the induced map $\mathcal{F}_{1.} \longrightarrow \mathcal{F}_{2.}$ is also a quasi-isomorphism. It implies that the functor $\mathbf{F}$ is full. Nevertheless it is not faithful. It can be seen as follows: let

$$
\mathcal{E}_.[-1] = \ldots \longrightarrow 0 \longrightarrow \underbrace{\mathcal{E}_.}_{1} \longrightarrow 0 \ldots
$$

and

$$
\mathcal{F}_. = \ldots \longrightarrow 0 \longrightarrow \underbrace{\mathcal{F}_.}_{0} \longrightarrow 0 \ldots
$$

be two complexes ($\mathcal{F}$ and $\mathcal{G}$ are vector bundles). $\mathrm{Hom}(\mathcal{F}_.,\mathcal{E}_.[1]) = \mathrm{Ext}^1(\mathcal{F},\mathcal{E})$. The map

$$
\mathrm{Hom}_{D^-(\mathrm{Coh}_X)}(\mathcal{F}_.,\mathcal{E}_.[1]) = \mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F},\mathcal{E}) \longrightarrow \mathrm{Ext}^1_{\tilde{\mathcal{O}}_X}(\tilde{\mathcal{F}},\tilde{\mathcal{E}}) = \mathrm{Hom}_{TC_X}(\mathbf{F}(\mathcal{F}),\mathbf{F}(\mathcal{G}))
$$

is not a monomorphism: $\mathrm{Ext}^1(\mathcal{O},\mathcal{F}) = H^1(\mathcal{F})$, but $H^1(X,\mathcal{F}) \neq H^1(\tilde{X},\tilde{\mathcal{F}})$. So our functor is not faithful. It proves our theorem.

Corollary 2.6 *In the notation as above let $\mathcal{E}$ and $\mathcal{G}$ be two vector bundles on X. Then the canonical map*

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}) \longrightarrow \mathrm{Ext}^1_{\tilde{\mathcal{O}}_X}(\tilde{\mathcal{F}}, \tilde{\mathcal{E}})$$

is surjective.

3 Coherent sheaves on a rational curve with one node

Let us consider first the case of the rational curve with one simple node. Suppose its equation is $zy^2 - x^3 - x^2 z = 0$. Then its normalization is $\tilde{X} = \mathbf{P}^1$. Suppose that the preimages of a singular point are $(0:1) = 0$ and $(1:0) = \infty$.
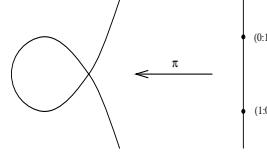
What does the result of the previous section mean? A complex $\mathcal{F}_{\cdot}$ from the derived category $D^-(\mathrm{Coh}_X)$ as a data structure is uniquely defined by some triple $(\tilde{\mathcal{F}}_{\cdot}, \mathcal{M}_{\cdot}, i)$.

What is $\tilde{\mathcal{F}}_{\cdot}$? The category $\mathrm{Coh}_{\mathbf{P}^1}$ has the global dimension 1. It means that (Dold, 1960) that indecomposable objects of $D^-(\mathrm{Coh}_{\mathbf{P}^1})$ are

$$\mathcal{E}_n[r] : \ldots \longrightarrow 0 \longrightarrow \underbrace{\mathcal{O}_{\mathbf{P}^1}(n)}_{r} \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots,$$

$$\mathcal{T}_{kx}[s] : \ldots \longrightarrow 0 \longrightarrow \mathcal{O}_{\mathbf{P}^1}(-kx) \hookrightarrow \underbrace{\mathcal{O}_{\mathbf{P}^1}}_{s} \longrightarrow 0 \longrightarrow \ldots.$$

A complex $\tilde{\mathcal{F}}_{\cdot}$ is just a direct sum

$$\tilde{\mathcal{F}}_{\cdot} \cong \bigoplus (\mathcal{E}_n[r] \oplus \mathcal{T}_{kx}[s]).$$

Now let us explain what is $\mathcal{M}_{\cdot}$ and what is i. $\mathcal{O}/\mathcal{J}$ is a skyscraper sheaf k_0 (with the stalk k at the singular point), $\tilde{\mathcal{O}}/\mathcal{J} = (k \times k)_0$. It means that the category $\mathrm{Coh}_{\mathcal{O}/\mathcal{J}}$ is semi-simple. So, $\mathcal{M}_{\cdot} \cong (H_{\cdot}(\mathcal{M}_{\cdot}), 0))$ and we get, moreover, a commutative diagram

$$
\begin{array}{ccc}
\mathcal{M}_{\cdot} & \xrightarrow{\ i\ } & \tilde{\mathcal{F}}_{\cdot}/\mathcal{J}) \\
\downarrow & & \downarrow \\
H_{\cdot}(\mathcal{M}_{\cdot}) & \xrightarrow{H_{\cdot}(i)} & H_{\cdot}(\tilde{\mathcal{F}}_{\cdot}/\mathcal{J})
\end{array}
$$

The map $H_k(i) : H_k(\mathcal{M}_{\cdot}) \longrightarrow H_k(\tilde{\mathcal{F}}_{\cdot}/\mathcal{J})$ is simply a map of two vector spaces. But $H_k(\tilde{\mathcal{F}}_{\cdot}/\mathcal{J})$ is also a $k \times k$-module. This implies that $H_k(i)$ is given

by two matrices $H_i(0)$ and $H_i(\infty)$ (intuitively — one corresponds to the point 0, another to ∞). Moreover, both of these matrices have the same size and are nondegenerated. It follows from the condition 3 of the category of triples: $H_k(\mathcal{M}.) \otimes_{\mathcal{O}/\mathcal{J}} \tilde{\mathcal{O}}/\mathcal{J} \xrightarrow{H_k(i) \otimes id} H_k(\tilde{\mathcal{F}}./\mathcal{J})$ is an isomorphism. Choose a set of local parameters of $\tilde{\mathcal{F}}$ in each component of the complex $\mathcal{F}.$. They induce some basis in $H.(\tilde{\mathcal{F}})$. Choose some basis in $H_k(\mathcal{M}.)$. With respect to such choice the map $H.(i)$ is given by a collection of matrices
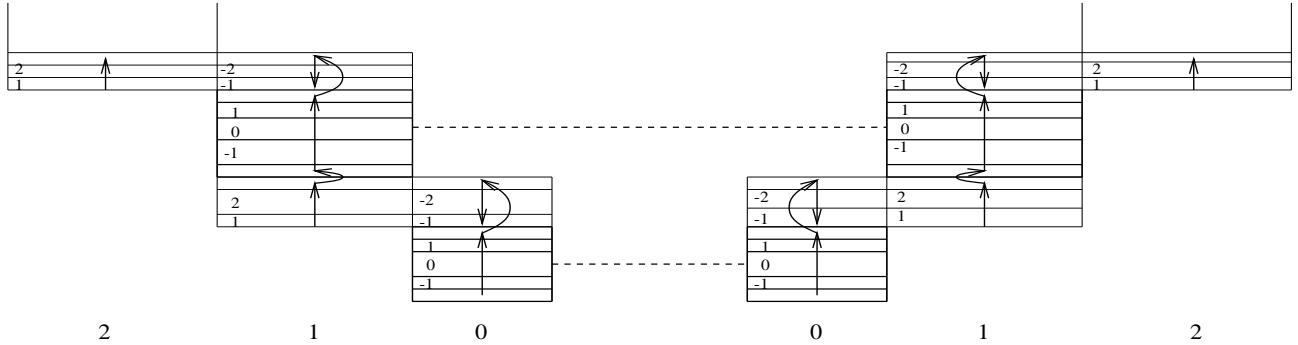
There are two types of blocks — those, which came from vector bundles and those which came from skyscraper sheaves. The blocks are numbered by integers and natural numbers, respectively. This numbering defines some "weights" of the blocks of vertical matrices (the partial order is shown in the picture). Blocks, corresponding to the same skyscraper or vector bundle, are called conjugated. Conjugated blocks have the same number of rows. Indeed, all the blocks of $H_i(0)$ and $H_i(\infty)$, but finitely many have the zero size. But if one of the conjugated blocks is nonempty, then the other one is nonempty, too.

Now we should answer on the following question: which triples $(\tilde{\mathcal{F}}., \mathcal{M}., H.(i))$ correspond to the isomorphic complexes $\mathcal{F}.$? Surely, we have to consider the automorphisms of $\tilde{\mathcal{F}}.$ and look at what they induce in homologies. As a result we get the following matrix problem:

1. we can do any simultaneous elementary transformations of the columns of the matrices $H_i(0)$ and $H_i(\infty)$;

2. we can do any simultaneous transformations of rows inside conjugated blocks;

3. we can add a scalar multiple of any row from the block with lower weight to any row of a block of the higher weight (inside of the big matrix, of course). These transformations can be proceeded independently inside of $H_i(0)$ and $H_i(\infty)$ (see the next section for more details).

These types of matrix problems are well-known in the representation theory. First they appeared in the work of Nazarova-Roiter ([12]) about the classification of $k[[x,y]]/(xy)$-modules. They are called, sometimes, Gelfand problems

in honour of I. M. Gelfand, who formulated a conjecture (at the International Congress of Mathematics in Nice (1970)) about the structure of Harish-Chandra modules at the singular point of $SL_2(\mathbb{R})$ [8]. This problem was reduced to some matrix problem of such a type [13].

The strict categorical formulation and then a solution of this type of problem was done by Nazarova-Roiter, Drozd, Bondarenko [13], [7], [4]. It means that these matrices correspond to the objects of some category and these objects are isomorphic if and only if one matrix can be transformed into another one by the above set of transformations. From this point of view, it is enough to describe the indecomposable objects.

Let us recall a combinatoric of the answer in this case. There are two types of indecomposable objects: bands and strings [3]. We shall give the definitions in the next section. Here we only want to stress, that band object depends on one continuous parameter and several discrete parameters. String object depands only on discrete parameters. Let us consider some examples.

Example 3.1 *The following data (band) define a simple vector bundle of rank 2 on X: normalization $\tilde{\mathcal{O}} \oplus \tilde{\mathcal{O}(n)}, n \neq 0$, matrices:*

Example 3.2 *The following data (band) define a mixed sheaf (sheaf which is neither torsion nor torsion-free): normalization is*

$$\mathcal{E}_0[0]^{2n} \oplus \mathcal{E}_{-3}[0]^n \oplus \mathcal{T}_{20}[0]^n \oplus \mathcal{T}_{50}[0]^n \oplus \mathcal{T}_{1\infty}[0]^n \oplus \mathcal{T}_{4\infty}[0]^n,$$

matrices are

Example 3.3 *The following data (string) define the skyscraper sheaf k_0: normalization is*

$$\bigoplus_{i=0}^{\infty}(\mathcal{T}_{10}[-i] \oplus \mathcal{T}_{1\infty}[-i]),$$

matrices:
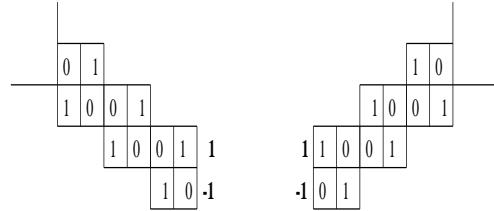
Example 3.4 *The following data (band) defines some object of the bounded category $D^b(Coh_X)$ which is not a coherent sheaf: normalization is*

$$(\mathcal{E}_{-2}[-2])^n \oplus (\mathcal{T}_{20}[-1])^n \oplus (\mathcal{T}_{1\infty}[-1])^n \oplus (\mathcal{E}_0[-1])^n \oplus (\mathcal{T}_{10})^{2n} \oplus (\mathcal{T}_{2\infty})^n \oplus (\mathcal{T}_{4\infty})^n \oplus (\mathcal{E}_{-1})^n \oplus (\mathcal{E}_1)^n$$

matrices:

Let us now consider a general case.
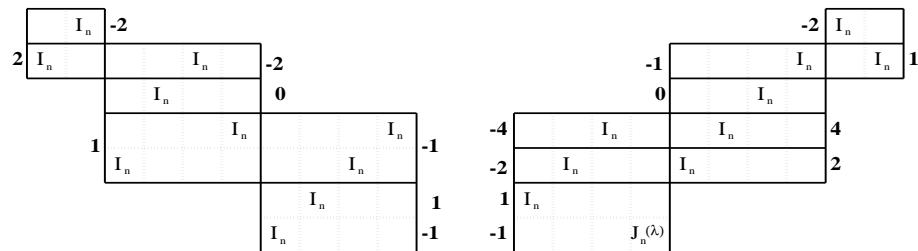
Example 3.5 *The following data (string) defines some object of the category $D^-(Coh_X)$ which is not an object of the bounded derived category $D^b(Coh_X)$: normalization is*

$$\cdots \oplus \mathcal{T}_{10}[-2] \oplus \mathcal{T}_{1\infty}[-2] \oplus \mathcal{E}_2[-2] \oplus \mathcal{T}_{30}[-1] \oplus \mathcal{T}_{1\infty}[-1] \oplus \mathcal{E}_{-1}[-1] \oplus \mathcal{T}_{20} \oplus \mathcal{T}_{3\infty} \oplus \mathcal{E}_0$$

matrices:
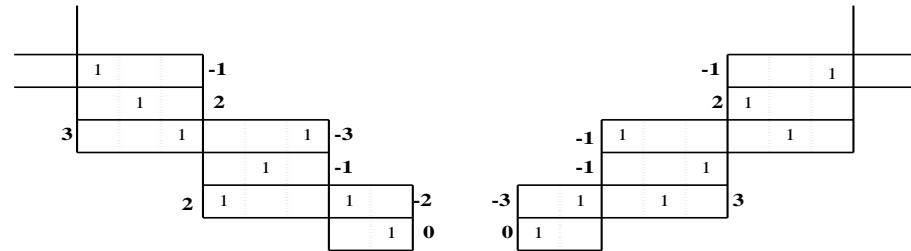
Let us now consider a general case.

4 Reduction to the matrix problem

Suppose X is either a rational curve with one node or a configuration of projective lines of the type A_n, $\tilde{A}_n$, $\tilde{X} \xrightarrow{\pi} X$ its normalization. Then $\tilde{X}$ is just a disjoint union of projective lines, $\tilde{X} = \coprod_{i=1}^{n} X_i$. Let π_i be the restriction of π on X_i, $\tilde{\mathcal{O}}_i = (\pi_i)_*(\pi^*(\mathcal{O}_X))$, $\mathcal{J}_i = \pi_*(\mathcal{J}|_{X_i})$. The category $\mathrm{Coh}_{\tilde{X}}$ is equivalent to $\prod_{i=1}^{n} \mathrm{Coh}_{X_i}$ (the same holds for the derived categories), $\tilde{\mathcal{O}} - \mathrm{mod}$ is equivalent to the direct product of the categories $(\tilde{\mathcal{O}}_1 - \mathrm{mod}) \times (\tilde{\mathcal{O}}_2 - \mathrm{mod}) \times \cdots \times (\tilde{\mathcal{O}}_n - \mathrm{mod})$. $\tilde{\mathcal{O}}/\mathcal{J} - \mathrm{mod}$ is equivalent to $(\tilde{\mathcal{O}}_1/\mathcal{J}_1 - \mathrm{mod}) \times (\tilde{\mathcal{O}}_2/\mathcal{J}_2 - \mathrm{mod}) \times \cdots \times (\tilde{\mathcal{O}}_n/\mathcal{J}_n - \mathrm{mod})$. Choose the local coordinates on each of the lines X_i in such a way that the preimages of the singular points are either $(0:1)$ or $(1:0)$. We can interpret coherent $\tilde{\mathcal{O}}_i$-modules just as $\mathrm{Coh}_{\mathbf{P^1}}$. $\mathcal{J}_i$ will be identified with the ideal sheaf of two points $(0:1)$ and $(1:0)$, or, if only one point lies on X_i, just with the ideal sheaf of $(0:1)$. Both $\mathcal{O}/\mathcal{J}$ and $\tilde{\mathcal{O}}/\mathcal{J}$ are the skyscraper sheaves with support in the singular points of X. Suppose X is the configuration of projective lines of the type $\tilde{A}_n$. The canonical morphism $\mathcal{O}/\mathcal{J} \longrightarrow \tilde{\mathcal{O}}/\mathcal{J}$ is then the diagonal morphism

$$k \times k \times \cdots \times k \longrightarrow (k \times k) \times (k \times k) \times \cdots \times (k \times k).$$

Let $(\tilde{\mathcal{F}}_\cdot, \mathcal{M}_\cdot, i)$ be some triple. $\tilde{\mathcal{F}}_\cdot \cong \tilde{\mathcal{F}}_{1\cdot} \oplus \tilde{\mathcal{F}}_{2\cdot} \oplus \cdots \oplus \tilde{\mathcal{F}}_{n\cdot}$, where $\tilde{\mathcal{F}}_{i\cdot} \in D^-(\mathrm{Coh}_{X_i}) = D^-(\mathrm{Coh}_{\mathbf{P^1}})$.

How do we get a matrix problem in this case? $\mathrm{Coh}_{\mathbf{P^1}}$ has a homological dimension 1, which implies that every indecomposable object of $D^-(\mathrm{Coh}_{\mathbf{P^1}})$ is isomorphic to some object of the type $\ldots \longrightarrow 0 \longrightarrow \underbrace{\mathcal{F}}_{i} \longrightarrow 0 \longrightarrow \ldots$,

where $\mathcal{F} \in Ob(\mathrm{Coh}_{\mathbf{P^1}})$ is indecomposable. Indecomposable objects of $\mathrm{Coh}_{\mathbf{P^1}}$ are known: line bundles $\mathcal{O}_{\mathbf{P^1}}(n)$ and skyscraper sheaves $\mathcal{O}_{\mathbf{P^1},x}/m_x^n$.

The skyscraper sheaf $\mathcal{O}_{\mathbf{P^1},x}/m_x^n$ has a locally free resolution

$$0 \longrightarrow \mathcal{O}_{\mathbf{P^1}}(-n) \longrightarrow \mathcal{O}_{\mathbf{P^1}} \longrightarrow \mathcal{O}_{\mathbf{P^1},x}/m_x^n \longrightarrow 0,$$

which means that in the derived category $(\mathcal{O}_{\mathbf{P^1}}(-n) \longrightarrow \mathcal{O}_{\mathbf{P^1}}) \cong \mathcal{O}_x/m_x^n$ holds.

Choose the local bases in each component of the complex $\tilde{\mathcal{F}}$. The map $H_k(i) : H_k(\mathcal{M}_\cdot) \longrightarrow H_k(\tilde{\mathcal{F}}_\cdot/\mathcal{J})$ is given by $n+1$ matrices, corresponding to the singular points of X. Each of these matrices itself consists of two nondegenerated components of the same size.

The morphisms in the derived category $D^-(\mathrm{Coh}_{\mathbf{P^1}})$ and their images after tensoring with $\tilde{\mathcal{O}}/\mathcal{J}$ and taking the homology, is very easy to compute in this case, so we just leave this computation.

The question is, which transformations can we do with the matrices defining the homology? From the definition of the category of triples follows that we have to consider the automorphisms $\Phi : \tilde{\mathcal{F}}_\cdot \longrightarrow \tilde{\mathcal{F}}_\cdot$ and $\phi : \mathcal{M}_\cdot \longrightarrow \mathcal{M}_\cdot$ which

12

make the following diagram

$$
\begin{array}{ccc}
H_.(\mathcal{M}_.) & \xrightarrow{\ H_.(i)\ } & H_.(\tilde{\mathcal{F}}_./\mathcal{J}) \\
\Big\downarrow{\scriptstyle H_.(\phi)} & & \Big\downarrow{\scriptstyle H_.(\Phi\otimes id)} \\
H_.(\mathcal{M}_.) & \xrightarrow{\ H_.(i)\ } & H_.(\tilde{\mathcal{F}}_./\mathcal{J})
\end{array}
$$

commutative.

With every curve from the list of Drozd-Greuel we can associate some partially ordered set. Let $\omega_{-1} < \omega_0 < \omega_1$ be three cardinal numbers (it means that $n\omega_{-1} < m\omega_0 < k\omega_1 \ \forall n, m, k \in \mathbb{Z}$). The algorithm is the following now: consider the set of pairs (L, a), where L is a component of $\tilde{X}$, $a \in L$ some preimage of the singular point. To such a pair we associate the following partially ordered sets: $F_{(L,a)}(k)$, $E_{(L,a)}(k)$ $(k \in \mathbb{N})$. $F_{(L,a)}(k)$ consist from one element. $E_{L,a}(0)$ has two types of elements $E_{(L,a)}(0, m\omega_{-1})$, $m \in \mathbb{Z}_-$ and $E_{(L,a)}(0, n\omega_0), n \in \mathbb{Z}_+$. $E_{(L,a)}(i)(i \geq 1)$ has three types of elements: $E_{(L,a)}(i, m\omega_{-1})(m \in \mathbb{Z}_-)$, $E_{(L,a)}(i, n\omega_0)(n \in \mathbb{Z})$, $E_{(L,a)}(i, k\omega_1)(k \in \mathbb{Z}_+)$. Each set $E_{(L,a)}(i)$ has a natural partial order.

Consider the union of all points

$$
\mathbf{E}\bigcup\mathbf{F} = (\ \bigcup_{L,a,i} E_{L,a}(i))\bigcup(\bigcup_{L,a,i} F_{L,a}(i)).
$$

In this set let us introduce the following equivalence relation:

1. $E_{(L,a)}(i, -n\omega_{-1}) \sim E_{(L,a)}(i+1, n\omega_1)$, $i \geq 0$;

2. $E_{(L,a)}(i, m\omega_0) \sim E_{(L',a)}(i, m\omega_0)$ $i \geq 0$;

3. $F_{(L,a)}(i) \sim F_{(L,a')}(i)$ $i \geq 0$.

Example 4.1 *Rational curve with one node*
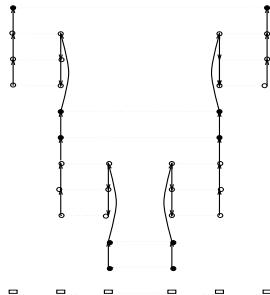
Example 4.2 *Transversal intersection of two lines at one point (A_1-case)*
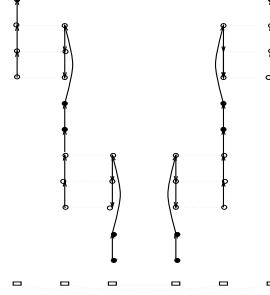
Example 4.3 *Transversal intersection of two lines at two points ($\tilde{A}_1$-case)*
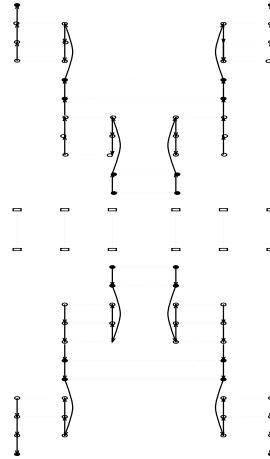
What we have is called a bunch of chains [3], which codes a matrix problem. The number of matrices can be infinite, but this does not disturb the general theory.

Namely, the matrix problem is the following:

1. Each triple (L, a, i), (L is a component of $\tilde{X}$, $a \in L$, $i \geq 0$ the integer number) corresponds to some matrix $M(L, a, i)$. These matrices are divided into horizontal blocks, numbered by the points of $E_{L,a}(i)$. Since $F_{L,a}$ consists only of one element, we do not have a vertical division in this case. Ideed, some blocks may have zero size, i.e., be empty.

2. Blocks, corresponding to the conjugated points from **E**, have the equal number of rows; points, corresponding to the conjugated points from **F**, have the equal number of columns

14

3. We have the partial order on the set of points. Let us say that the horisontal blocks are supplied with some "weights", and the weight of one block is bigger than the weight of another block if the point, corresponding to the first block, is bigger than the point corresponding to the second one.

4. We can do the following transformation with our matrices:

   (a) Simultaneous: elementary transformations with the columns of matrices $M(L, a, i)$ and $M(L', a, i)$.

   (b) Simultaneous: any elementary transformations inside of the conjugated blocks.

   (c) Independent: add a scalar multiple of any row from the block with lower weight to any row of a block of the higher weight.

In our case there are some additional restrictions on our matrices.

1. All big matrices are square and nondegenerated.

2. If one of the conjugated blocks is nonempty, that the other one is nonempty, too.

There are two types of indecomposable objects: bands and strings.

1. Band data $\mathcal{B}(w, m, \lambda)$ is given by two discrete parameters: by word w, natural number m, and one continuous parameter $\lambda \in k^*$. A word w is just a sequence of points of $\mathbf{E} \cup \mathbf{F}$ $w_1 - w_2 \sim w_3 - w_4 \sim \cdots - w_N$, connected by the symbols of two types, $-$ and $\sim$. The symbol $\sim$ should stay between conjugated points, $-$ only between a point of the type $E_{L,a}(*, i)$ and a point $F_{L,a}(i)$. If one link was $-$, then the next one should be $\sim$ and vice versa. In a band data a word w should be closed: $x_N \sim x_1$. It means that it can be written as a cycle. We require that w is not a power of some other word.

2. A string data $\mathcal{T}(w)$ depends only on some full word w. Full means that w contains each point w_i together with its conjugate. In our situation a word w could be infinite. We require, however, that each point w_i appears only a finite number of times.

Let us briefly recall the algorithm, giving a concrete description of matrices, corresponding to the band and string data.

1. Let a band data be $\mathcal{B}(w, m, \lambda)$. We count the entrance of each class of conjugated points. Let point w_i occured k_i times. The block, corresponding to w_i, should be divided into k_i strips. We have a division of a big matrix $M(L, a, i)$ into the smaller blocks. Let us look now at the subwords $w_i - w_{i+1}$. Suppose we have the k-th appearance of the class $[w_i]$ and l-th appearance of the class $[w_{i+1}]$. One of the points w_i, w_{i+1} belongs to $\mathbf{E}$,

15

the other to **F**. If $w_{i+1} \neq w_N$, then we put in the entry with the coordinates (k, l) (with respect to the subpartition of $w_i \times w_{i+1}$ submatrix) the identity matrix I_m (here our second discrete parameter appears). If $w_{i+1} = w_N$, then we put, on the corresponding place, the Jordan block $J_m(\lambda)$. All other entries are zero.

2. Let a string data be $\mathcal{T}(w)$. The algorithm is basically the same as in the case of bands. The only difference is that we have to put not the I_m or $J_m(\lambda)$, but just 1×1 matrix 1.

It is clear that the closed word is necessarily finite. A word, defining a string data, could be infinite.

Consider an example. We write down the data, corresponding to the complexes on the nodal rational curve, which were considered in the previous chapter.

Example 4.4 *Let* $w = \overset{1}{E}_{(X,a)}(0, -1\omega_0) - F_{(X,a)}(0) \overset{1}{\sim} F_{(X,b)}(0) - E_{(X,b)}(0, 1\omega_0) \overset{1}{\sim}$
$E_{(X,a)}(0, 1\omega_0) - F_{(X,a)}(0) \overset{2}{\sim} F_{(X,b)}(0) - E_{(X,b)}(0, -2\omega_1) \sim E_{(X,b)}(1, 2\omega_{-1}) -$
$F_{(X,b)}(1) \overset{1}{\sim} F_{(X,a)}(1) - E_{(X,a)}(1, \omega_{-1}) \overset{1}{\sim} E_{(X,a)}(0, -\omega_1) - F_{(X,a)}(0) \overset{3}{\sim} F_{(X,b)}(0) -$
$E_{(X,b)}(0, -4\omega_1) \overset{1}{\sim} E_{(X,b)}(1, 4\omega_{-1}) - F_{(X,b)}(1) \overset{2}{\sim} F_{(X,a)}(1) - E_{(X,a)}(1, 0\omega_0) \overset{1}{\sim}$
$E_{(X,b)}(1, 0\omega_0) - F_{(X,b)}(1) \overset{3}{\sim} F_{(X,a)}(1) - E_{(X,a)}(1, -2\omega_{-1}) \overset{1}{\sim} E_{(X,a)}(2, 2\omega_{-1}) -$
$F_{(X,a)}(2) \overset{1}{\sim} F_{(X,b)}(2) - E_{(X,b)}(2, -2\omega_0) \overset{1}{\sim} E_{X,a}(2, -2\omega_0) - F_{(X,a)}(2) \overset{2}{\sim} F_{(X,b)}(2) -$
$E_{(X,b)}(2, 1\omega_{-1}) \overset{1}{\sim} E_{(X,b)}(1, -1\omega_1) - F_{(X,b)}(1) \overset{4}{\sim} F_{(X,a)}(1) - E_{(X,a)}(1, 1\omega_{-1}) \overset{2}{\sim}$
$E_{(X,a)}(0, -1\omega_{-1}) - F_{(X,a)}(0) \overset{4}{\sim} F_{(X,b)}(0) - E_{(X,b)}(0, -\omega_0) \overset{1}{\sim}$. *Then* $\mathcal{B}(w, n, \lambda)$
is just a complex from Example 3.4 from the previous chapter.

Theorem 4.5 *(See [3].)*

1. *All representations* $\mathcal{B}(\omega, m, \lambda)$, $\mathcal{S}(\omega)$ *are indecomposable. Each indecomposable representation is isomorphic, either to some band representation* $\mathcal{B}(\omega, m, \lambda)$ *or to some string representation* $\mathcal{S}(\omega)$.

2. *The only isomorphisms between these objects are*

   (a) $\mathcal{S}(\omega) \cong \mathcal{S}(\omega^{-1})$, *where* $\omega = a_0 r_1 a_1 \ldots r_m a_m$ *and* $\omega^{-1} = a_m r_m a_{m-1} \ldots r_1 a_0$ *is the inverse word.*

   (b) $\mathcal{B}(\omega, m, \lambda) = \mathcal{B}(\omega', m, \lambda')$, *where* ω' *is a cyclic permutation of* ω *and* λ' *depending on the signum of the permutation being either* λ *or* λ^{-1}.

We want now to illustrate the convenience of our description of the complexes in the derived category $D^-(\mathrm{Coh}_X)$. Let $\mathcal{F}_\cdot$ and $\mathcal{G}_\cdot$ be two objects, given by triples $(\tilde{\mathcal{F}}_\cdot, \mathcal{M}_\cdot, i)$ and $(\tilde{\mathcal{G}}_\cdot, \mathcal{N}_\cdot, j)$. We can ask which triple corresponds to the tensor product of complexes $\mathcal{F}_\cdot \otimes_L \mathcal{G}_\cdot$. As one can easily see, it should be $(\tilde{\mathcal{F}}_\cdot \otimes \tilde{\mathcal{G}}_\cdot, \mathcal{M}_\cdot \otimes \mathcal{N}_\cdot, i \otimes j)$.

By the Künneth formula we have a functorial isomorphism (since the homological dimension is 0):

$$\bigoplus_{k+l=n} (H_k(\mathcal{M}_{\cdot}) \otimes (H_l(\mathcal{N}_{\cdot}) \xrightarrow{\oplus(H_k(i)\otimes H_l(j))} H_n(\mathcal{M}_{\cdot} \otimes \mathcal{N}_{\cdot}).$$

This means that we can compute the matrices, corresponding to the tensor product of complexes.

5 Description of the coherent sheaves, vector bundles, torsion-free sheaves, mixed sheaves and skyscraper sheaves

Now we want to show what corresponds to coherent sheaves. Let complex $\mathcal{F}_{\cdot}$ be given by a triple $(\tilde{\mathcal{F}}_{\cdot}, \mathcal{M}_{\cdot}, i)$. We have to write the conditions $H_i(\mathcal{F}_{\cdot}) = 0, (i \geq 1)$ in the language of matrices. Recall that we have the following diagram:

$$
\begin{array}{ccccccc}
\mathcal{J}\tilde{\mathcal{F}}_{\cdot} & \longrightarrow & \tilde{\mathcal{F}}_{\cdot} & \longrightarrow & \tilde{\mathcal{F}}_{\cdot} \otimes_{\tilde{\mathcal{O}}} \tilde{\mathcal{O}}/\mathcal{J} & \longrightarrow & \mathcal{J}\tilde{\mathcal{F}}_{\cdot}[-1] \\
{\scriptstyle id}\uparrow & & {\scriptstyle \Phi}\uparrow & & {\scriptstyle i}\uparrow & & {\scriptstyle id}\uparrow \\
\mathcal{J}\tilde{\mathcal{F}}_{\cdot} & \longrightarrow & \mathcal{F}_{\cdot} & \longrightarrow & \mathcal{M}_{\cdot} & \longrightarrow & \mathcal{J}\tilde{\mathcal{F}}_{\cdot}[-1].
\end{array}
$$

Write the long exact sequence of homologies, associated with this morphism of triangles:

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & H_0(\tilde{\mathcal{F}}_{\cdot}/\mathcal{J}\tilde{\mathcal{F}}_{\cdot}) & \longleftarrow & H_0(\tilde{\mathcal{F}}_{\cdot}) & \longleftarrow & H_0(\mathcal{J}\tilde{\mathcal{F}}_{\cdot}) & \longleftarrow & H_1(\tilde{\mathcal{F}}_{\cdot}/\mathcal{J}\tilde{\mathcal{F}}_{\cdot}) & \longleftarrow & \cdots \\
 & & {\scriptstyle H_0(i)}\uparrow & & \uparrow & & {\scriptstyle id}\uparrow & & {\scriptstyle H_1(i)}\uparrow \\
0 & \longleftarrow & H_0(\mathcal{M}_{\cdot}) & \longleftarrow & H_0(\mathcal{F}_{\cdot}) & \longleftarrow & H_0(\mathcal{J}\tilde{\mathcal{F}}_{\cdot}) & \longleftarrow & H_1(\mathcal{M}_{\cdot}) & \longleftarrow & \cdots
\end{array}
$$

We get $H_i(\mathcal{F}_{\cdot}) = 0, (i \geq 1)$ is equivalent to

1. $H_1(\mathcal{M}_{\cdot}) \longrightarrow H_1(\tilde{\mathcal{F}}_{\cdot}/\mathcal{J}\tilde{\mathcal{F}}_{\cdot}) \longrightarrow H_0(\mathcal{J}\tilde{\mathcal{F}}_{\cdot})$ being a monomorphism.

2. $H_{k+1}(\mathcal{M}_{\cdot}) \longrightarrow H_{k+1}(\tilde{\mathcal{F}}_{\cdot}/\mathcal{J}\tilde{\mathcal{F}}_{\cdot}) \longrightarrow H_k(\mathcal{J}\tilde{\mathcal{F}}_{\cdot})$ being an isomorphism.

Let us give a combinatorical interpretation of these conditions (for both bands and strings). Let w be a parameter either of $\mathcal{B}(w, m, \lambda)$ or of $\mathcal{S}(\omega)$. These conditions imply:

1. A word w does not contain any $E_{L,a}(k+1, n\omega_0), k \geq 0$;

2. a word w does not contain any $E_{L,a}(k+1, n\omega_{-1})(n \geq 1, k \geq 1)$, $E_{L,a}(k+1, -m\omega_1), (m \geq 2, k \geq 1)$;

3. a word w does not contain any subword of type $E_{L,a}(k+1, \omega_1) - F_{L,a}(k+1) \sim F_{L',a}(k+1) - E_{L',a}(k+1, \omega_1)$.

It gives us the description of coherent sheaves:

1. All the bands $\mathcal{B}(w, m, \lambda)$ such that the word w doesn't contain points $F_{L,a}(i)$ with $i \geq 2$.

2. Strings $\mathcal{T}(w)$ with the following properties:

   (a) There are no points $E_{L,a}(n\omega_0, i)$ with $i \geq 2$, $E_{L,a}(k+1, n\omega_{-1})$, $E_{L,a}(k+1, -m\omega_1)$ with $n, m \geq 2$.

   (b) w does not contain any subword of type $E_{L,a}(k+1, \omega_1) - F_{L,a}(k+1) \sim F_{L',a}(k+1) - E_{L',a}(k+1, \omega_1)$.

   (c) For each $i \geq 0$ the points of the type $E_{L,a}(*, i)$ appeared the same number of times as $F_{L,a}(i)$ (it is a condition for the matrices $M(L, a, i)$ to be square and nondegenerated).

In a similar way we can describe the *bounded* derived category $D^b(Coh_X)$: the conditions $H_i(\mathcal{F}), i \gg 0$ can be described in a similar way.

In particular we get a description of

1. Vector bundles (we get just the matrix problem from the work of Drozd and Greuel): bands $\mathcal{B}(w, m, \lambda)$ with w not containing $F_{L,a}(i), (i \geq 1)$ (in case of a curve of arithmetic genus 1).

2. Skyscraper sheaves: bands and strings, defining a coherent sheaf and not containing $E_{L,a}(n\omega_0, i), i \geq 0$ (it follows from the observation that $\mathcal{F}$ and $\mathcal{F} \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$ have the same support).

We see that for a coherent sheaf $\mathcal{F}$ we have either $\mathcal{T}or_{\mathcal{O}}^i(\mathcal{F}, \mathcal{O}/\mathcal{J}) = 0$ for $i > 1$ or it is nonzero for all $i \geq 2$. As a corollary we obtain that the homological dimension of an object of Coh_X is either 0 or 1 or ∞ (which coincides with the result of the Auslander-Buchsbaum formula).

We are going to do the last step in our classification: we shall describe among all the coherent sheaves, the torsion-free sheaves. Those which are not vector bundles have infinite homological dimension, and hence we should look for them among strings. Let $\mathcal{F}$ be a coherent sheaf on X. It is torsion-free if and only if all its localizations $\mathcal{F}_x$ are torsion-free $\mathcal{O}_{X,x}$-modules. At the regular point this condition is obvious. But we go further, $\mathcal{F}_x$ is a torsion-free $\mathcal{O}_{X,x}$-module if and only if its completion $\hat{\mathcal{F}}_x$ is a torsion-free $\hat{\mathcal{O}}_{X,x}$-module. But in our case, if x is singular, then $\hat{\mathcal{O}}_{X,x} = k[[x, y]]/(xy)$. The indecomposable torsion-free modules are known in this case, they are either $k[[x]]$ or $k[[y]]$ or the regular module $k[[x, y]]/(xy)$ [2].

Now, let us mention that in the same way we have dealt with curves, we can deal with the local ring $k[[x, y]]/(xy)$. Namely, we consider its normalization $k[[x]] \times k[[y]]$, conductor $J = (x, y)$ and just repeat the construction of the category of triples. As a result, we get the following matrix problem (in the notation of Bondarenko):
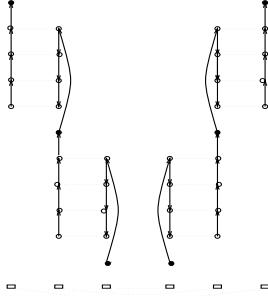
Let $x \in X$ be singular. Consider the functor $\mathrm{Coh}_X \longrightarrow (\mathcal{O}_{X,x} - \mathrm{mod}) \longrightarrow (k[[x,y]]/(xy) - \mathrm{mod})$ (composition of the localization and completion). This functor is exact and so induces the functor between the derived categories $D^-(\mathrm{Coh}_X) \longrightarrow D^-(k[[x,y]]/(xy) - \mathrm{mod})$. What does it look like on triples? Obviously, $(\tilde{\mathcal{F}}_{\cdot\cdot}, \mathcal{M}_{\cdot}, i)$ is mapped to $\widehat{(\tilde{\mathcal{F}}_{\cdot})}_x, \widehat{(\mathcal{M}_{\cdot})}_x, i_x)$. So, the image of the triple is described by the same matrices! But surely there is one important difference: blocks, corresponding to vector bundles are united, and there are no links between them anymore. But we know how the modules $k[[x]]$, $k[[y]]$ and $k[[x,y]]/(x,y)$ are given in the language of triples. Let us denote $T_{nx} : k[[x]] \xrightarrow{x^n} k[[x]], T_{ny} : k[[y]] \xrightarrow{y^n} k[[y]]$. Then $k[[x]]$, for example, is given by normalization:

$$k[[x]] \oplus (\bigoplus_{i=0}^{\infty} T_{iy}[-i] \oplus T_{i1x}[-i-1])$$

and matrices:
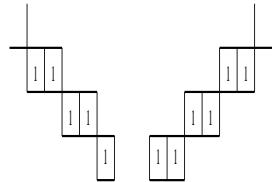
Hence we can deduce the answer: torsion-free sheaves, which are not vector bundles, are strings $\mathcal{T}(w)$, where w does not contain any $E_{L,a}(i, n\omega_{-1})$, $E_{L,a}(i, -m\omega_1)$ ($i \geq 2, m, n \geq 2$). Moreover, each $E_{L,a}(i, \omega_{-1})$, $E_{L,a}(i, -\omega_1)$ can occur in word w at most one time.

6 The construction of Polishchuk

In a recent paper A. Polishchuk [14] showed a connection between the structure of the derived category of the coherent sheaves $D^b(\mathrm{Coh}_X)$, where X is a projective curve of arithmetical genus 1 with nodal singularities and the tri-

honometric solutions of the classical Yang-Baxter equation. The main role in this construction is played by the so called spherical objects:

Definition 6.1 *(See [15].) Let $\mathcal{D}$ be a triangulated category over a field k, such that all spaces $\mathrm{Hom}(X, Y)$ are finite-dimensional. $\mathrm{Hom}^i(M, N) := \mathrm{Hom}(M, N[i])$. An object M is called n-spherical, if*

1. *$\mathrm{Hom}^i(M, M) = 0 \ \forall i \neq 0, n, \ \mathrm{Hom}^0(M, M) \cong \mathrm{Hom}^n(M, M) \cong k$.*

2. *$\forall F \in Ob(\mathcal{D})$ the composition map $\mathrm{Hom}^i(M, F) \times \mathrm{Hom}^{n-i}(F, M) \longrightarrow \mathrm{Hom}^n(M, M) \cong k$ is nondegenerated.*

Let X be a projective curve as above. $\mathcal{D} = D^b(Coh_X)$. It was shown in [14] that all simple vector bundles are 1-spherical. Indeed, the first condition follows from the theorem of Riemann-Roch for singular curves. Namely, let $\mathcal{E}$ be a simple vector bundle. This implies that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) \cong k$. We have to show that $\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \cong k$. From the local-global spectral sequence follows that

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{E}) \cong H^1(\mathcal{H}om(\mathcal{E}, \mathcal{E})) \cong H^1(\mathcal{E}^\vee \otimes \mathcal{E}).$$

$\mathcal{E}^\vee \otimes \mathcal{E}$ is a vector bundle. By the theorem of Riemann-Roch $\chi(\mathcal{E}^\vee \otimes \mathcal{E}) = deg(\mathcal{E}^\vee \otimes \mathcal{E}) + 1 - p_a(X) = 0$ holds. Hence $\dim_k(\mathrm{Ext}^1(\mathcal{E}, \mathcal{E})) = \dim_k(\mathrm{Hom}(\mathcal{E}, \mathcal{E})) = 1$.

The second condition follows from the Serre duality for the derived categories (see for example [15]). One can also show that the skyscraper sheaves $\mathcal{O}_x/m_x$ (x ist regular) are also 1-spherical.

Conjecture 6.2 *(See [14].) What are the 1-spherical objects in this case? Which orbits have the set of spherical objects under the action of the group of autoeqivalences of the derived category $Aut(\mathcal{D})$?*

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